EE120 - Fall'19 - Lecture 3 Notes[1]

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5 September 2019

Response of LTI Systems to Complex Exponentials

Section 3.2 *in* Oppenheim & Willsky

Complex Exponentials

Continuous-time:

$$x(t) = e^{st},\ s \in \mathbb{C} \xrightarrow{s=\sigma+j\omega} x(t) = \underbrace{e^{\sigma t}}_{\text{envelope}}\ \underbrace{e^{j\omega t}}_{\text{periodic}} \tag{1}$$

Discrete-time:

$$x[n] = z^n,\ z \in \mathbb{C} \xrightarrow{z=re^{j\omega}} x[n] = r^n e^{j\omega n} \tag{2}$$

Figures 1 and 2 on page 5 plot e^{st} and z^n for various values of s and z in the complex plane.

The response of a LTI system to a complex exponential is the same complex exponential scaled by a constant.

$$e^{st} \rightarrow \boxed{h(t)} \rightarrow y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = \underbrace{\left(\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau\right)}_{\triangleq H(s)} e^{st} \tag{3}$$

$$z^n \rightarrow \boxed{h[n]} \rightarrow y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = \underbrace{\left(\sum_{-\infty}^{\infty} h[k]z^{-k}\right)}_{\triangleq H(z)} z^n \tag{4}$$

$H(s)$ and $H(z)$ are called "transfer functions" or "system functions."

Example: Find the transfer function $H(s)$ for $y(t) = x(t-3)$.
If $x(t) = e^{st}$ then

$$y(t) = x(t-3) = e^{s(t-3)} = \underbrace{e^{-3s}}_{=H(s)} e^{st}. \tag{5}$$

Alternatively, use the impulse response $h(t) = \delta(t-3)$:

$$H(s) = \int_{-\infty}^{\infty} \delta(\tau-3)e^{-s\tau}d\tau = e^{-3s}. \tag{6}$$

Frequency Response of a LTI System

$$\begin{aligned} x(t) = e^{j\omega t} (s = j\omega) &\quad\rightarrow\quad y(t) = H(j\omega) e^{j\omega t} \\ x[n] = e^{j\omega n} (z = e^{j\omega}) &\quad\rightarrow\quad y[n] = H(e^{j\omega}) e^{j\omega n} \end{aligned} \tag{7}$$

$$\boxed{\begin{aligned} H(j\omega) &= \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau \\ H(e^{j\omega}) &= \sum h[k] e^{-j\omega k} \end{aligned}} \tag{8}$$

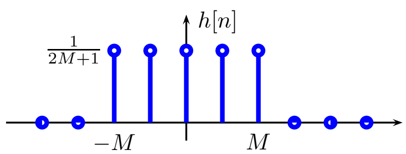
Filtering

Section 3.9 *in* Oppenheim & Willsky

LTI system designed such that $H(j\omega)$ ($H(e^{j\omega})$ in DT) is zero or close to zero for frequencies to be eliminated.

Example: Why is the moving average system a low-pass filter?

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{M} x[n-k] \tag{9}$$

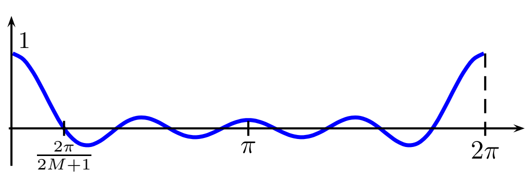


$$H(e^{j\omega}) = \sum_{k=-M}^{M} \frac{1}{2M+1} e^{-j\omega k} = \frac{e^{j\omega M}}{2M+1} \underbrace{\left(1 + e^{-j\omega} + ... + e^{-j\omega 2M}\right)}_{\frac{1-e^{-j\omega(2M+1)}}{1-e^{-j\omega}} \text{ if } w \neq 0^2}$$

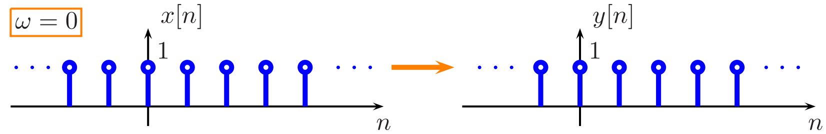
$$= \frac{1}{2M+1} \underbrace{e^{j\omega M} \frac{e^{-j\omega(M+\frac{1}{2})}}{e^{-j\omega/2}}}_{=1} \underbrace{\frac{e^{j\omega(M+\frac{1}{2})} - e^{-j\omega(M+\frac{1}{2})}}{e^{j\omega/2} - e^{-j\omega/2}}}_{\frac{\sin(\omega(M+1/2))}{\sin(\omega/2)}}$$

[2] since this is a geometric series of the form $1 + x + \cdots + x^{2M}$, which equals $\frac{1-x^{2M+1}}{1-x}$ when $x \neq 1$. Substitute $x = e^{-j\omega}$ and note that $x \neq 1$ means $\omega \neq 0$.

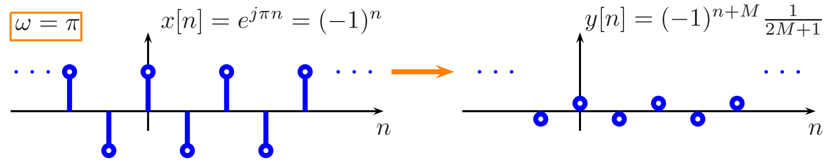
$$H(e^{j\omega}) = \begin{cases} 1 & \text{if } \omega = 0 \\ \frac{1}{2M+1} \frac{\sin(\omega(M+1/2))}{\sin(\omega/2)} & \omega \neq 0 \end{cases} \tag{10}$$



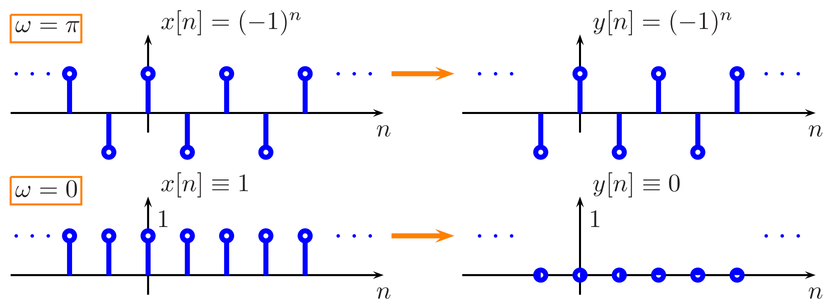
Low frequencies pass through:



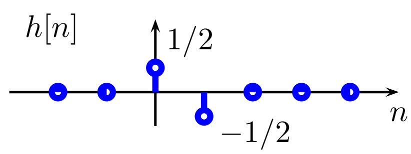
High frequencies are attenuated:



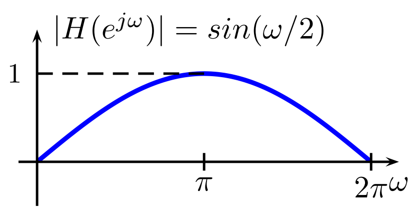
Example: Is $y[n] = \frac{1}{2}(x[n] - x[n-1])$ low-pass or high-pass?



To find $H(e^{j\omega})$, note that the impulse response is:



$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \frac{1}{2} - \frac{1}{2}e^{-j\omega} = \frac{1}{2}(1 - e^{-j\omega}) = \frac{1}{2}e^{-j\omega/2}2jsin(\omega/2) \tag{11}$$



FIR vs. IIR Systems

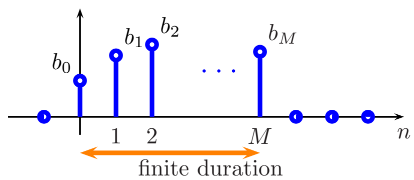
Note that the examples above have impulse responses of finite duration. Such systems are called Finite Impulse Response (FIR) systems.

A causal *finite impulse response* (FIR) system has the form:

$$y[n] = b_0 x[n] + ... + b_M x[n-M] \tag{12}$$

and its impulse response is:

$$h[n] = b_0 \delta[n] + b_1 \delta[n-1] + ... + b_M \delta[n-M]. \tag{13}$$

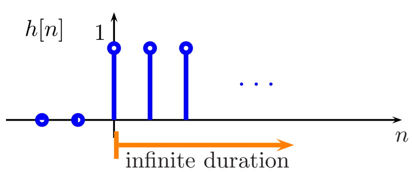


Note that a FIR system is always stable, because the sum $\sum_n |h[n]|$ is over a finite duration and, thus, finite.

An *infinite impulse response* (IIR) example:

$y[n] - y[n-1] = x[n]$, $y[-1] = 0$, $x[n] = 0$ for $n < 0$ (accumulator)

Impulse response: $h[n] = u[n]$



Constant-coefficient linear difference equations like those above (and differential equations in continuous time) are a rich source of LTI systems. The general form of a constant-coefficient linear difference equation is:

$$\boxed{a_0 y[n] + a_1 y[n-1] + ... + a_N y[n-N] = b_0 x[n] + ... b_M x[n-M]} \tag{14}$$

which is causal and LTI if $a_0 \neq 0$ and the system is "initially at rest" (that is, $y[n] = 0$ for $n < n_0$, where $n = n_0$ is the first instant when $x[n] \neq 0$). Recall that a linear system must return a zero output in response to a zero input, and this property is destroyed when the initial conditions are non-zero.

We usually take $a_0 = 1$, since otherwise we can divide all coefficients by a_0 and normalize the coefficient of $y[n]$ to one. Thus, we can implement the system (14) with the recurrence relation:

$$y[n] = -a_1 y[n-1] - ... - a_N y[n-N] + b_0 x[n] + ... b_M x[n-M].$$

When $a_1 = \cdots = a_N = 0$ the difference equation (14) reduces to the FIR system (12). Therefore, the source of IIR behavior is the presence of feedback terms $-a_1 y[n-1] - ... - a_N y[n-N]$.

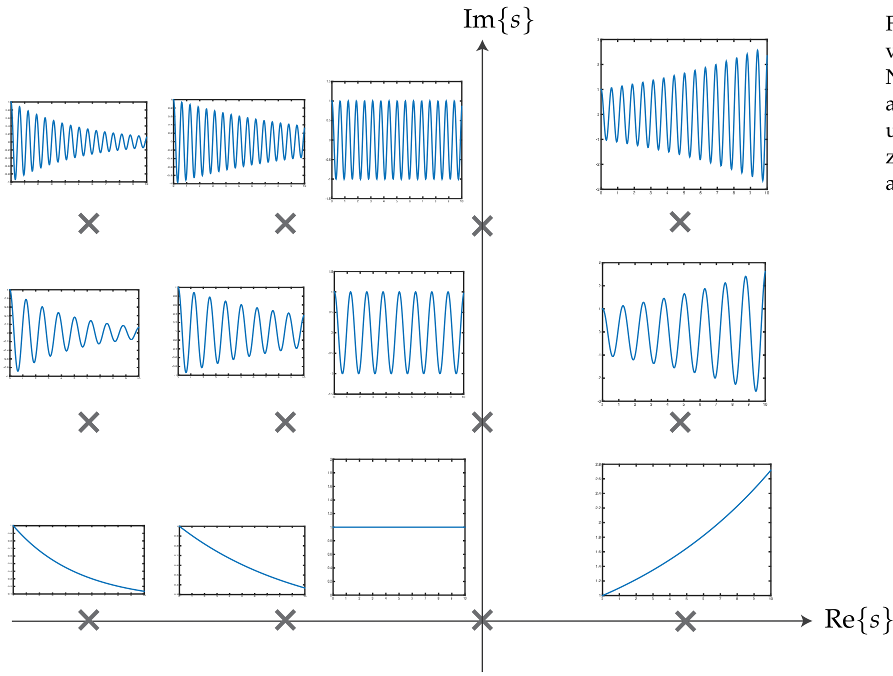


Figure 1: The real part of e^{st} for various values of s in the complex plane. Note that e^{st} is oscillatory when s has an imaginary component. It grows unbounded when $\mathrm{Re}\{s\} > 0$, decays to zero when $\mathrm{Re}\{s\} < 0$, and has constant amplitude when $\mathrm{Re}\{s\} = 0$.

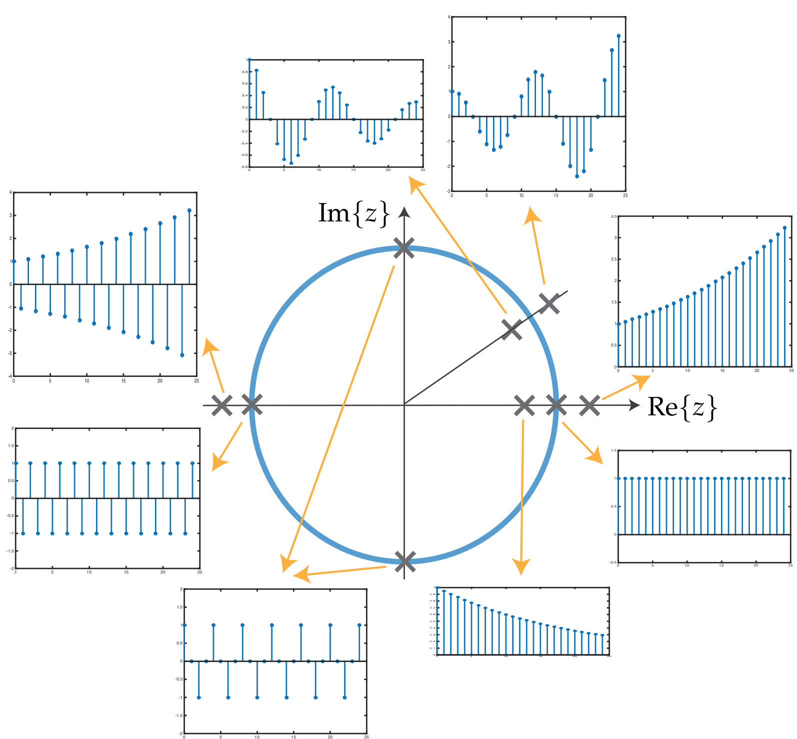


Figure 2: The real part of z^n for various values of z in the complex plane. It grows unbounded when $|z| > 1$, decays to zero when $|z| < 1$, and has constant amplitude when z is on the unit circle ($|z| = 1$).